Time-Optimal solutions of Parallel Navigation and Finsler geodesics

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Abstract

A geometric approach to kinematics in control theory is illustrated. A non-linear control system is derived for the problem and the Pontryagin maximum principle is used to find the time-optimal trajectories of the Parallel navigation. The time-optimal trajectories of the Parallel navigation are characterized through a geometric formulation. It is notable that the approach has the advantages using feedback.[1]

Keywords: Finsler geometry, Parallel navigation, Kinematics, Optimal control, Pontryagin maximum principle.

1 Introduction

The historical development of what became the Calculus of Variations is closely linked to certain minimization principles in the majority subjects in mechanics, namely, *the principle of least distance*, *the principle of least time* and ultimately, *the principle of least action* [7]. To understand solution of the well-known *brachistochrone* problem, (i.e finding a curve from point A to point B along which a free-sliding particle will descend more quickly than on any other AB-curve), we are led through *Fermat's principle of least time*: light always takes a path that minimizes travel time.

The *Parallel navigation*, or briefly P-navigation, is a quiet old problem and has been studied using several techniques from the viewpoints of kinematics and dynamics in optimal control theory [17]. The application of Finsler geometry in Physics, seismology and Biology is a subject of numerous papers such as [1], [2],[3], [5], [9], [13], [15], [18], etc. Let O be the origin of an inertial reference frame of coordinates (FOC). The positions of M and T in this (FOC) are given

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by the vectors $\mathbf{r}_M = OM$ and $\mathbf{r}_T = OT$, respectively. In two-point guidance systems, the vector $\mathbf{r} = \mathbf{r}_T - \mathbf{r}_M$ is conventionally called the *range*. Its time derivative $\dot{\mathbf{r}} = \dot{\mathbf{r}}_T - \dot{\mathbf{r}}_M = \mathbf{v}_T - \mathbf{v}_M$ is *the relative velocity* between the two objects, and $\mathbf{v}_T$ and $\mathbf{v}_M$ are the velocities of T and M, respectively. W always denote the vectors by bold face and their norms will be shown by the same normal letter. As an application, it is notable for mariners wishing to rendez-vous each other at sea. M could be a boat and T, a tanker with fuel for it (or vice-versa). Or, back in history, T could be a merchantman and M a pirate ship. This rule assumes, of course, constant speeds. Thus, in most realistic cases, v_T and v_M are supposed to be constant. However, it is easy to extend the theory if they are not constant. The *closing velocity*, a term often used in the study of guidance, is simply $\mathbf{v}_C = -\dot{\mathbf{r}}$. Notice that, we wish to study the kinematics of P-navigation in a relative (FOC) rather than a absolute one, i.e., we shall seek the location of M in a (FOC) attached to T. Thus, a trajectory in the relative (FOC) shows the situation as seen by an observer located at T. As the special cases, we assume that $M = R^3$ or $M = R^2$. In reality, the velocity $\mathbf{v}_T$ and $\mathbf{r}_T$ can be detected and reported at any $\mathbf{r}$ by a grounded radar. Suppose that $\delta(\mathbf{r})$ be the angle between $\mathbf{v}_M$ and MT and given any δ, there is Finsler metric F given by:

$$F(\mathbf{r}, \mathbf{v}, \delta) = \frac{|\mathbf{v}|^2}{v_M \cos\delta |\mathbf{v}| - \langle \mathbf{v}, \mathbf{v}_T \rangle}, \tag{1}$$

where, $|.|$ denotes the Riemannian norm on M. A *solution* of the described P-navigation is a curve $(\mathbf{r}(t), \delta(t))$ such that respects the required constraints on velocities.

Theorem 1.1 *Given any solution* $(\mathbf{r}, \delta)$ *of parallel navigation, the curve* $\mathbf{r}$ *can be reparametrized so that it satisfies* $F(\mathbf{r}(t), \mathbf{v}(t), \delta(t)) = 1$.

The *indicatrix* $S(\mathbf{r}, \delta)$ of the metric (1) is the set of unit tangent vectors $\mathbf{v}$ with respect to (1) which is defined by $S(\mathbf{r}, \delta) = \{\mathbf{v} \in T_{\mathbf{r}}M \mid F(\mathbf{r}, \mathbf{v}, \delta) = 1\}$. Following Theorem 1.1, at any time t we have $\dot{\mathbf{r}} = \mathbf{v} \in S(\mathbf{r}, \delta)$. Hence, at any time t, there is a unit vector $f(\mathbf{r}, \delta) \in S(\mathbf{r}, \delta)$ such that $\dot{\mathbf{r}} = \mathbf{v} = f(\mathbf{r}, \delta)$.
Control problems typically concern finding a (not necessarily unique) control law $\delta(.)$, which transfers the system in finite time from a given initial state $x_i = \mathbf{r}(0)$, to a given final state $x_f = \mathbf{r}(t_f)$. This transition is to occur along an admissible path, i.e. $\mathbf{r}(.)$ and respects all kinematic constraints imposed on it. Let us consider it as

$$\dot{\mathbf{r}} = f(\mathbf{r}, \delta). \tag{2}$$

We further assume that $\delta(.)$ is admissible, i.e. is piecewise continuous and belongs to $\mathcal{U}$, the admissible control space. Let there now be a rule which assigns a unique, real-valued number to each of these transfers. Such a rule can be viewed as the transition cost between x_i and x_f along an admissible path, completely specified by $\delta(.)$. The Optimal control concerns specifying this rule and thereby providing a systematic method for selecting the best, or optimal control law, according to some prescribed cost functional. One can find an analogue discussion in [5], to calculate the travel-time along the trajectories of

the so called *Pure pursuit navigation.* Here, the P-navigation optimal control problem can be founded by the cost function $C(\mathbf{r},\delta) = F(\mathbf{r},\dot{\mathbf{r}},\delta)$ and has the following form

$$\text{minimize} \int_0^{t_f} C(\mathbf{r},\delta)dt, \tag{3}$$

where, $t_f \in (0,\infty)$ is the final time which is going to be optimized. From everyday experience we know that collision courses need not be straight lines if T changes its speed or direction; so what is exactly the collision course? It may be curved in some sense. One of our goal in this paper is to make known the best collision course.

Theorem 1.2 *Given any time-optimal solution* $(\mathbf{r},\delta)$ *of P-navigation, the curve* $\mathbf{r}$ *is a geodesic of the Finsler metric (1).*

The trajectory $\mathbf{r}_M$ can be obtained $\mathbf{r}_M = \mathbf{r}_T - \mathbf{r}$ when $\mathbf{r}$ is known. One can freely consider $\mathbf{v}_M$ and $\mathbf{v}_T$ as vector fields alon $\mathbf{r}$. Now, let $\frac{\nabla}{dt}$ be the covariant derivative defined for any vector field Y along $\mathbf{r}$ defined by

$$\frac{\nabla Y^i}{dt} := \frac{dY^i}{dt} + G^i_{jk}(\mathbf{r},\dot{\mathbf{r}},\delta)Y^jY^k,$$

where, G^i_{jk} are the connection coefficients of Berwald connection associated to the Finsler metric (1). As a result of Theorem 1.2, we can mention the following result:

Theorem 1.3 *The time-optimal trajectory* $\mathbf{r}_M$ *of P-navigation satisfies the following second order ODE:*

$$\ddot{\mathbf{r}}^i_M + G^i_{jk}(\mathbf{r},\mathbf{v},\delta)\mathbf{v}^j_M\mathbf{v}^k_M = \frac{\nabla \mathbf{v}^i_T}{dt}, \qquad i = 1,...,n.$$

Our approach is closely related with subjects such as non-holonomic mechanics, sub-Finslerian geometries, see for a deeper sight [8] and [4]. One may find various techniques in missile guidance and control in [17].

2 Preliminaries

Let M be a n-dimensional C^∞ manifold. T_xM denotes the tangent space of M at x. The tangent bundle of M is the union of tangent spaces $TM := \cup_{x\in M}T_xM$. We will denote the elements of TM by (x,y) where $y \in T_xM$. Let $TM_0 = TM \setminus \{0\}$. The natural projection $\pi : TM_0 \to M$ is given by $\pi(x,y) := x$.
A *Finsler metric* on M is a function $F : TM \to [0,\infty)$ with the following properties; (i) F is C^∞ on TM_0, (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, and (iii) the y-Hessian of $\frac{1}{2}F^2$ with elements $g_{ij}(x,y) := \frac{1}{2}[F^2(x,y)]_{y^iy^j}$ is positive definite on TM_0. The pair (M,F) is then called a *Finsler space.* The Riemannian metrics are special Finsler metrics. Traditionally, a Riemannian metric is denoted by $a_{ij}(x)dx^i \otimes dx^j$. It is a family

of inner products on tangent spaces. Let $\alpha(x,y) := \sqrt{g_{ij}(x)y^i y^j}$, $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$. α is a family of Euclidean norms on tangent spaces. Throughout this paper, we also denote a Riemannian metric by $\alpha = \sqrt{a_{ij}(x)y^i y^j}$.

An (α,β)-metric is a scalar function on TM defined by $F := \Phi(\frac{\beta}{\alpha})\alpha$, where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$ with certain regularity. $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on a manifold M. One may find another important class of (α,β)-metrics in [16]. The *Randers* and *Matsumoto* metrics are special (α,β)-metrics defined by $\Phi = 1+s$ and $\Phi = \frac{1}{1-s}$, respectively, i.e, $F = \alpha + \beta$ and $F = \frac{\alpha^2}{\alpha-\beta}$. Randers metrics were introduced by Randers in 1941 [13] in the context of general relativity. In [6], applying Fermat's principle, the authors proved that the time-optimal solutions of the well-known Zermelo's navigation-moving that is the motion of a vehicle equipped with an engine with a fixed power output in presence of a wind current-are actually the geodesics of a Randers metric. M. Matsumoto gave an exact formulation of a Finsler surface to measuring the time on the slope of a hill and introduced the Matsumoto metrics in [9], see also [15].

A Lagrangian on the manifold M is a mapping $L : TM \longrightarrow R$ which is smooth on TM_0. A Lagrangian is said to be *regular* if it has non-degenerate y-Hessian on TM_0. Thus, given a Finsler metric F, the function $L = \frac{F^2}{2}$ is a regular Lagrangian. A large area of applicability of this geometry is suggested by the connections to Biology, Mechanics, and Physics and also by its general setting as a generalization of Finsler and Riemannian geometries [10]. For every smooth curve $c : [a,b] \longrightarrow R$, the extremal curves of the action integral given by

$$I(c) = \int_a^b L(c(t), \dot{c}(t))dt, \tag{4}$$

are characterized locally by the *Euler-Lagrange equations* given as follows:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0, \tag{5}$$

where, $x^i(t)$ is a local coordinate expression of c. The extremal curves of the action integral (4) are usually called *the geodesics of* L. In [1] it is shown that the Lagrangian and Finslerian approaches are projectively the same.

Given a Finsler manifold (M,F), a globally defined vector field G is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$, where $G^i(x,y)$ are local functions on TM_0 satisfying $G^i(x,\lambda y) = \lambda^2 G^i(x,y)$, $\lambda > 0$, see [14]. G is called the associated *spray* to (M,F). In local coordinates, a curve $c(t)$ is a geodesic of F if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(c,\dot{c}) = 0$.

2.1 The kinematics of Parallel navigation

We shall refer to the target as T and to the pursuer as M and their velocities as v_M and v_T, respectively. To begin, we set up a coordinate system called

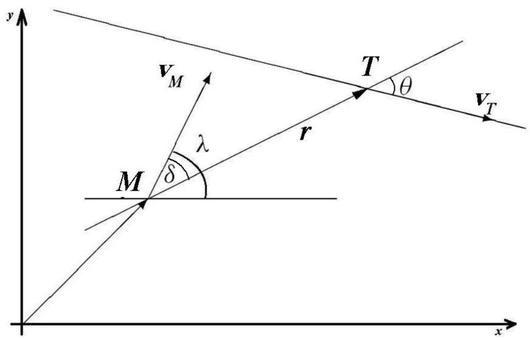

Figure 1: *The range* $\mathbf{r}$*, the velocity vectors* $\mathbf{v}_M$ *and* $\mathbf{v}_T$*.*

reference frame of coordinates, in which the pursuer is initially located at the origin O. When considering planar motion we shall use Cartesian coordinates (x, y) or (x, z), and the angles will be positive if measured counterclockwise. The ray that starts at the pursuer M and is directed at the target T along the positive sense of $\mathbf{r}$ is called *the line of sight* (LOS). The *parallel navigation* geometrical rule,has been known since antiquity, mostly by mariners. According to this rule, the direction of the line of sight, MT, is kept constant relative to inertial space, i.e., the LOS is kept parallel to the initial LOS. In three-dimensional vector terminology, the rule is very concisely stated as $\mathbf{r} \times \dot{\mathbf{r}} = 0$. Suppose that θ and λ denote, respectively, the angles between $\mathbf{v}_T$ and $\mathbf{v}_M$ and, $\mathbf{v}_M$ and the horizontal axis (Figure 1).

Let us put $r = |\mathbf{r}|$. The basic rule for moving of the pursuer is presented by the following two equations [17]:

$$\dot{r} = v_T \cos\theta - v_M \cos\delta, \tag{6}$$

$$r\dot{\lambda} = v_T \sin\theta - v_M \sin\delta. \tag{7}$$

Notice that, in a planar framework, $\mathbf{v}_M$, $\mathbf{v}_T$ and $\mathbf{r}$ being on the same (fixed) plane by definition, therefore, the parallel navigation geometrical rule can be restated as $\dot{\lambda} = 0$. The requirement $\langle \mathbf{r}, \mathbf{v} \rangle < 0$ must be added in order to ensure that M should approach T not recede from it. In this case, we have $\dot{r} < 0$, that is $v_T \cos\theta < v_M \cos\delta$. Let us denote the projection of any vector $\mathbf{v}_T$ on $\mathbf{v}$ by $Proj_{\mathbf{v}}\mathbf{v}_T$. A *solution* of the described P-navigation is a curve $(\mathbf{r}(t), \delta(t))$ such that respects the equations (6) and (7). By *the trajectory* of P-navigation, we mean a curve $\mathbf{r}(t)$ such that $(\mathbf{r}(t), \delta(t))$ is a solution, for some control δ.

Initiating the process, we have $\mathbf{r}(0) = \mathbf{r}_0$ which shows that, M stands at a point with distance r_0 from T. Through the performance, r decreases by time and hence, M approaches T. Therefore, $\mathbf{r}$ tends to the origin O and M hits T when $\mathbf{r}(t_f) = 0$, (Figure 2). It follows that, P-navigation trajectories are characterized by a curve $\mathbf{r}$ joining $Q = \mathbf{r}_0$ to the origin O (Figure 3). It is of our interests to find the best QO-trajectory. More precisely, the problem is *to find a curve from point Q to point O along which a particle will descend more*

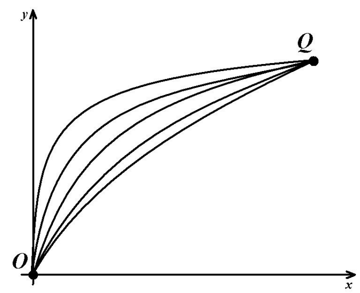

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Figure 2: *Some possible ranges initiated at the point Q.*

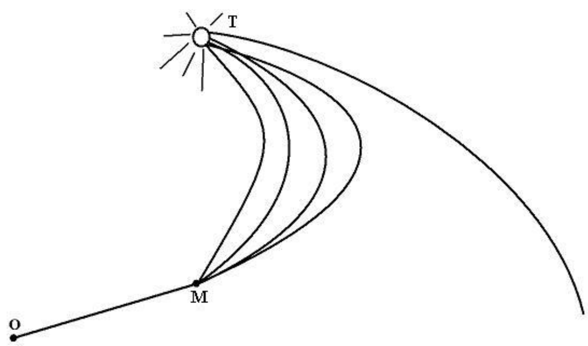

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Figure 3: *Schematic of exemplary collision courses for M.*

quickly than on any other QO-curve of P-navigation. In this way, the problem somehow resembles to a brachistochrone problem.

3 The optimal control theory.

A *control system* of ordinary differential equations is a family of differential equations in normal form $\frac{d\mathbf{r}^i}{dt} = f^i(\mathbf{r}, \delta)$, where $\mathbf{r}^i$ are called *state variables*, t is the *parameter of evolution* (usually the time) and δ^a are the *controls*. Geometrically, it can be regarded as a fibred mapping $X : U \longrightarrow TM$, from a control fiber bundle (U, η, M) over the state manifold M to the tangent bundle (TM, π, M), see [11]. Using local coordinates $(\mathbf{r}^i)$, $i = 1, ..., n$ in M, adapted coordinates $(\mathbf{r}^i, \delta^a)$, $a = 1, ..., k$ in U, and natural coordinates $(\mathbf{r}^i, \mathbf{v}^i)$ in TM, the coordinate expression for X is $X(\mathbf{r}, \delta) = f^i(\mathbf{r}, \delta)\frac{\partial}{\partial \mathbf{r}^i}$, or $\mathbf{v}^i = f^i(\mathbf{r}, \delta)$, the family of control equations. *Admissible* curves of the control system are curves $\gamma : I \subset R \longrightarrow U$ such that $(\eta o \gamma)^c = X o \gamma$, where ^{c} denotes the natural lifting to TM of a curve in M. Interested readers are advised to see [11] for getting familiar to the geometry of control systems. In Optimal Control Theory, a *cost functional* $\mathcal{C}(\gamma) = \int C(\mathbf{r}(t), \delta(t))dt$ is given and the goal is to

obtain admissible curves of the control system, satisfying some boundary conditions (e.g. $x_i = \mathbf{r}(0)$, $x_f = \mathbf{r}(t_f)$) and minimizing the cost functional. It is therefore a Classical Variational problem with non-integrable constraints defined by the control equations. Pontryagin maximum principle [12] provides a set of necessary conditions for a solution $(\mathbf{r}(t), \hat{\delta}(t))$ to be optimal; introducing a Hamiltonian function

$$\begin{aligned} H(\mathbf{r}, \mathbf{p}, \delta) &:= \langle \mathbf{p}, X \rangle - C(\mathbf{r}, \delta) = \mathbf{p}_i f^i(\mathbf{r}, \delta) - C(\mathbf{r}, \delta), \\ \hat{H}(\mathbf{r}, \mathbf{p}) &:= \max_{\delta} H(\mathbf{r}, \mathbf{p}, \delta). \end{aligned}$$

where the variables $(\mathbf{p}_i)$ are momenta coordinates, the optimal curves $(\mathbf{r}(t), \hat{\delta}(t))$ must satisfy the control system equations

$$\mathbf{v}^i = \frac{\partial \hat{H}}{\partial \mathbf{p}^i} = f^i(\mathbf{r}(t), \hat{\delta}(t))$$

and there must exist a solution curve for the adjoint differential equations

$$\frac{d\mathbf{p}_i}{dt} = -\frac{\partial \hat{H}}{\partial \mathbf{r}^i},$$

Define the Lagrangian L by $L(\mathbf{r}, \mathbf{v}) = \mathbf{p}_i \mathbf{v}^i - \hat{H}$. Observe that we have the following relations

$$\frac{d\mathbf{r}}{dt} = \frac{\partial \hat{H}}{\partial \mathbf{p}} = \mathbf{v}, \qquad \frac{d\mathbf{p}}{dt} = -\frac{\partial \hat{H}}{\partial \mathbf{r}} = \frac{\partial L}{\partial \mathbf{r}}, \qquad \frac{\partial \hat{H}}{\partial \mathbf{v}} = \mathbf{p} - \frac{\partial L}{\partial \mathbf{v}} = 0.$$

From the above equations, it results the well-known Euler-Lagrange for L

$$\frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}} - \frac{\partial L}{\partial \mathbf{r}} = 0.$$

Proposition 3.1 *[12] In order for $(\mathbf{r}(t), \hat{\delta}(t))$ to be an optimal solution of (3), the following are necessary conditions:*
(a) There exists a solution curve for the adjoint differential equations

$$\frac{d\mathbf{p}_i}{dt} = -\frac{\partial \hat{H}}{\partial \mathbf{r}^i}.$$

(b) $\hat{\delta} = \arg \max_{\delta} H(\mathbf{r}, \mathbf{p}, \delta), \quad \forall t \in [0, t_f].$
(c) $\hat{H}(\mathbf{r}, \mathbf{p}) = 0, \quad \forall t \in [0, t_f].$

4 Proof of Theorems.

4.1 Proof of Theorem 1.1

Let $(\mathbf{r}(t), \delta(t))$ be a pair of the curve $\mathbf{r}$ and a function $\delta(t)$. We are going to show that, if $(\mathbf{r}(t), \delta(t))$ be a solution of P-navigation, then $\mathbf{t}(t)$ must be

reparametrized so that we we have $F(\mathbf{r}(t), \dot{\mathbf{r}}(t), \delta(t)) = 1$. We notice that, in P-navigation, $\mathbf{r}$ and $\mathbf{v}$ are collinear and $\dot{r} < 0$, hence we have

$$\dot{r} = \frac{\langle \mathbf{r}, \mathbf{v} \rangle}{r} = \pm |Proj_{\mathbf{r}} \mathbf{v}| = \pm |Proj_{\mathbf{v}} \mathbf{v}| = -|\mathbf{v}|.$$

Now, we summarize (6) in the following relation

$$|\mathbf{v}| = v_M \cos \delta - \frac{\langle \mathbf{v}_T, \mathbf{v} \rangle}{|\mathbf{v}|}.$$

After simplification, we obtain the following equation

$$F(\mathbf{r}, \mathbf{v}, \delta) = \frac{|\mathbf{v}|^2}{v_M \cos \delta |\mathbf{v}| - \langle \mathbf{v}_T, \mathbf{v} \rangle} = 1.$$

Q.E.D.

4.2 Proof of Theorem 1.2

Following Theorem 1.1, at any time t we have $\dot{\mathbf{r}} = \mathbf{v} \in S(\mathbf{r}, \delta)$. Hence, at any time t, there is a unit vector $X(\mathbf{r}, \delta) \in S(\mathbf{r}, \delta)$ such that $\dot{\mathbf{r}} = \mathbf{v} = X(\mathbf{r}, \delta)$. Consider the unit canonical vector field $\ell(\mathbf{r}, \dot{\mathbf{r}}, \delta) = \frac{\dot{\mathbf{r}}}{F(\mathbf{r}, \dot{\mathbf{r}}, \delta)}$. We notice that, in P-navigation framework, we always assume that $\mathbf{r}$ and $\dot{\mathbf{r}}$ are collinear and hence, one can understand ℓ as a function of $\mathbf{r}$ and δ, as well. It follows that, given any trajectory $\mathbf{r}$ of P-navigation, X is given by $X(\mathbf{r}, \delta) = \ell(\mathbf{r}, \dot{\mathbf{r}}, \delta)$. Therefore, it is clear that,

$$\begin{aligned} \langle \mathbf{p}, X \rangle &= p_i f^i(\mathbf{r}, \delta) = p_i \ell^i(\mathbf{r}, \dot{\mathbf{r}}, \delta) = F(\mathbf{r}, \dot{\mathbf{r}}, \delta), \\ \langle \mathbf{p}, \mathbf{v} \rangle &= p_i \mathbf{v}^i = F^2(\mathbf{r}, \dot{\mathbf{r}}, \delta). \end{aligned}$$

Now, we return to the control system of P-navigation given by (2) with the cost functional $C(\mathbf{r}, \delta) = F(\mathbf{r}, \dot{\mathbf{r}}, \delta)$. It is easy to verify that, $H = 0$, $\hat{H} = 0$ and one may consider $\hat{\delta}$ as any possible control law. The conditions of Proposition 3.1 holds as well and the Lagrangian $L_{\hat{\delta}} = \langle \mathbf{p}, \mathbf{v} \rangle - \hat{H}$ is obtained as

$$L_{\hat{\delta}}(\mathbf{r}, \dot{\mathbf{r}}) = F^2(\mathbf{r}, \dot{\mathbf{r}}, \hat{\delta}).$$

Therefore, based on Pontryagin maximum principle, the optimal trajectories $\mathbf{r}(t)$ are geodesics of the Lagrangian $L_{\hat{\delta}}$. Clearly, they are geodesics of the Finsler metric $F(\mathbf{r}, \dot{\mathbf{r}}, \delta)$.

Now, consider the control-parametric family of Finsler metrics defined by $F_\delta(\mathbf{r}, \dot{\mathbf{r}}) := F(\mathbf{r}, \dot{\mathbf{r}}, \delta)$. Let $\mathcal{L}_\delta(\gamma) = \int_0^{t_f} F_\delta(\gamma, \dot{\gamma}) dt$ be the length of any admissible curve $\gamma(t)$ on (M, F_δ). A simple calculation gives the following inequality:

$$F_0(\mathbf{r}, \dot{\mathbf{r}}) \leq F_\delta(\mathbf{r}, \dot{\mathbf{r}}), \quad \text{for all possible controls } \delta.$$

From that, it follows that the functional $\mathcal{L}_\delta(\gamma)$ takes its minimum at $\delta = 0$, that is

$$\mathcal{L}_0(\gamma) \leq \mathcal{L}_\delta(\gamma), \quad \text{for all possible controls } \delta.$$

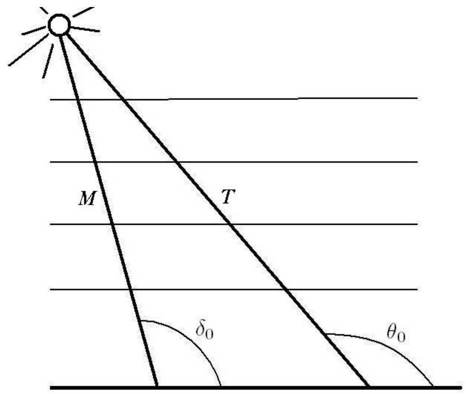

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Figure 4: *Collision course for a target moving on a straight line at a direction with a constant angle* θ_0.

Therefore, to find a time-optimal solution, one should minimize the cost functional $\mathcal{C}(\gamma) = \int F_0(\gamma, \dot{\gamma})dt$ and this leads us to obtain it as a geodesic of F_0. Q.E.D.

Theorem 4.1 *The time-optimal trajectory of P-navigation is a geodesic* $\mathbf{r}(t)$ *of the Finsler metric* $F_0 = \frac{|\mathbf{v}|^2}{v_M|\mathbf{v}| - \langle \mathbf{v}_T, \mathbf{v}\rangle}$.

However, given any control law, one may obtain a geodesic of the metric F_δ as the time-optimal trajectory. As a remark, we quote that the target T may not be reachable by the control $\delta = 0$.

Example 4.1 *(Case of plane nonmaneuvering target.) The target* T *is said to be nonmaneuvering if* $\mathbf{a}_T = 0$. *In this case,* T *moves on a straghit line at velocity* v_T *in the direction with a constant angle* θ_0 *if measured counterclockwise, see Figure 4. Let us suppose* $\mathbf{v}_T(x^1, x^2) = v_T\{\cos\theta_0 \frac{\partial}{\partial x^1} + \sin\theta_0 \frac{\partial}{\partial x^2}\}$. *Thus, from (7), it follows that* $\delta = \sin^{-1}(\frac{\sin\theta_0}{K})$, *where,* K *is the velocity ratio* $K = \frac{v_M}{v_T}$. *Then,* δ *is a constant say* δ_0. *Moreover,* $\mathbf{v}_T$ *is a parallel vector field and then* F_δ *is a Minkowski metric and is flat. Thus, it geodesics are straight lines. We obtain* $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}_0$. *But, from (6), we have* $|\mathbf{v}| = |\mathbf{v}_0| = v_M \cos\delta_0 - v_T \cos\theta_0$. *Intercept occur when we have* $\mathbf{r}(t_f) = 0$, *thus, the total flight time* t_f *is obtained by*

$$t_f = \frac{r_0}{v_M \cos\delta_0 - v_T \cos\theta_0} = \frac{r_0}{v_T(K\cos\delta_0 - \cos\theta_0)}$$

and the total range of M *equals* r_0 *which is the shortest curve joining* $\mathbf{r}_0$ *to the origin* O.

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